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001

EXISTENCE OF SOLUTIONS TO THE ELASTOHYDRODYNAMICAL EQUATIONS FOR MAGNETIC RECORDING SYSTEMS*

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Abstract. The existence of steady-state solutions to the system of nonlinear partial differential equations which are used to model the elastohydrodynamics of magnetic recording systems is demonstrated.

Key words. elastohydrodynamics, Reynolds lubrication equation, nonlinear partial differential equations

AMS(MOS) subject classifications. 35J65, 73J06

1. Introduction. The purpose of this paper is to demonstrate the existence of steady-state solutions under appropriate conditions to the system of nonlinear partial differential equations which are used to model the elastohydrodynamics of magnetic recording systems. There are two components to these mechanical systems: a medium such as a disk pressure which develops in the air bearing between the medium and the recording head causes a deflection in the medium and since the deflection of the medium influences the pressure in the air bearing.

For simplicity, we shall restrict our attention to disk systems. Let $\Omega \subset \mathbb{R}^2$ be the annular region of the disk

$$\Omega = \{ x = (x_1, x_2) \mid R < r < 1 \}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and let $\Gamma \subset \Omega$ be the region where the head is in close proximity to the disk. Thus, we have scaled the spatial variables by the outer radius of the disk. The mathematical model that we use for the transverse displacement of the disk, $u = u(x,t)$, is given by [8], [13]

$$\rho\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial \theta}\right)^2 u = \nabla \cdot (T\nabla u) - \frac{E_p t_p^3}{12(1-\nu^2)}\Delta^2 u - \gamma\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial \theta}\right) u + p - p_a, \tag{1.1}$$

$$x = (x_1, x_2) \in \Omega, \qquad -\infty < t < \infty,$$

where t is time, θ is the angular coordinate in polar coordinates, ρ is area density, ω is the angular speed of rotation of the disk, T is tension, $E_p > 0$ is Young's modulus, t_p is the disk thickness, ν is Poisson's ratio, $\gamma > 0$ is the air damp coefficient, $p = p(x,t)$ is the pressure developed in the air bearing, and p_a is the ambient pressure.

It is reported in [13] that "earlier work has shown that by bonding tensioned flexible recording media to rigid support disks it is possible to have performance features

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similar to rigid disks, while retaining the advantages of flexible-media technology." For tensioned flexible recording media, the imposed tension is a scalar constant and the centrifugal tension is insignificant [13]. Thus, we shall take the total tension, T, to be a scalar constant. Since tensioned flexible recording media are bonded to rigid support disks at the inner and outer edge of the media (the support disk rotates with the medium), the appropriate boundary conditions are that the disk is clamped at the edges,

$$u = 0, \qquad r = R, 1, \tag{1.2}$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = 0, \qquad r = R, 1, \tag{1.3}$$

where n is the exterior normal to Ω. More details about tensioned flexible recording media and their advantages with respect to rigid (hard) disks and floppy disks are reported in [11].

The pressure, $p = p(x,t)$, is obtained from the compressible Reynolds lubrication equation [3]–[5], [8], [13]

$$\begin{aligned} &12\mu\frac{\partial(ph)}{\partial t} + 6\mu\mathbf{V}\cdot\nabla(ph) = \nabla\cdot(h^3p\nabla p), \qquad && x = (x_1, x_2) \in \Gamma, \\ &p = p_a, && x = (x_1, x_2) \in \partial\Gamma, \end{aligned} \tag{1.4}$$

where μ is the dynamic viscosity of the air, $h = h(x,t)$ is the thickness of the fluid layer between the head and the disk, and $\mathbf{V} = \mathbf{V}(x) = \omega(-x_2, x_1)$ is the velocity of the disk. We extend p to $\Omega - \Gamma$ by $p \equiv p_a$. If $\varphi = \varphi(x)$ represents the transverse coordinate of the head, then

$$h = h(u) = \varphi - u\,. \tag{1.5}$$

It will be convenient to define the dependent variables as functions of Cartesian coordinates, $x = (x_1, x_2)$, and as functions of radial coordinates, (r, θ), in different parts of this paper. However, we will denote the pressure, for example, by $p = p(x,t)$ or by $p = p(r,\theta,t)$. It will always be clear from the context which representation is appropriate.

Since $h = h(u) = \varphi - u$, the system (1.1) and (1.4) is a highly nonlinear, coupled system of partial differential equations. The physical problem requires that the variables be constrained by $p \geq 0$ and $h \geq 0$ ($h < 0$ would mean that a "head crash" has occurred). In this paper, we demonstrate the existence of a steady-state solution to (1.1)–(1.4) provided the parameters satisfy a given inequality. It is not known if there always exists a steady-state solution to the elastohydrodynamical system (1.1) and (1.4). Further, it is not known in general when unique asymptotically stable steady-state solutions exist to the elastohydrodynamical system. We note, though, that it has been demonstrated in [12] that steady-state solutions to (1.1) are asymptotically stable.

These questions of existence, uniqueness, and asymptotic stability for realistic parameter values are of practical interest in the design of magnetic recording systems. "Steady-state" solutions are often found numerically by integrating the time-dependent equations, and it can be difficult to determine whether we have converged to a steady-state, a slowly varying transient, or a slowly varying periodic solution. In this paper, we show that steady-state solutions do exist for appropriate material constants, design parameters, and operating conditions.

In §2, we shall give some estimates for the steady-state of (1.1). In §3, we shall review the estimates that we have obtained for (1.4) in [5], and we shall give the analysis for the existence of steady-state solutions for the coupled system (1.1) and (1.4). Applications to floppy disk systems and tape systems are also given in §§2 and 3.

We suppose the reader familiar with the usual Sobolev spaces $H^1(\Omega)$, $H^k(\Omega)$, and $H_0^k(\Omega)$, and we refer to [1] for details and notation.

2. The steady-state for the deflection of the rotating disk. In this section, we shall analyze the following steady-state equation for (1.1) to find $u \in H_0^2(\Omega) \cap H^4(\Omega)$ such that

$$\rho\omega^2 \frac{\partial^2 u}{\partial \theta^2} = \nabla \cdot (T \nabla u) - E\Delta^2 u - \gamma\omega \frac{\partial u}{\partial \theta} + p, \qquad x \in \Omega, \tag{2.1}$$

where

$$E \equiv \frac{E_p t_p^3}{12(1-\nu^2)} \tag{2.2}$$

and $p \in L^2(\Omega)$.

Since $p = p(r,\theta)$ is mean square integrable, we have the Fourier expansion

$$p(r,\theta) = \sum_{m=-\infty}^{+\infty} p_m(r) e^{im\theta} \tag{2.3}$$

where the coefficients are given by

$$p_m(r) = \frac{1}{2\pi} \int_0^{2\pi} p(r,\theta) e^{-im\theta}\, d\theta. \tag{2.4}$$

It follows from the orthogonality of $e^{im\theta}$ that

$$\sum_{m=-\infty}^{+\infty} 2\pi \int_R^1 |p_m(r)|^2 r\, dr = \int_0^{2\pi} \int_R^1 |p(r,\theta)|^2 r\, dr\, d\theta < \infty$$

and, hence,

$$\int_R^1 |p_m(r)|^2 r\, dr < \infty.$$

We also assume the solution $u = u(r,\theta)$ to be mean square integrable, and we similarly expand

$$u(r,\theta) = \sum_{m=-\infty}^{+\infty} u_m(r) e^{im\theta}$$

where

$$u_m(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta) e^{-im\theta}\, d\theta.$$

We again have that

$$\int_0^{2\pi} \int_R^1 |u(r,\theta)|^2 r\, dr\, d\theta = \sum_{m=-\infty}^{+\infty} 2\pi \int_R^1 |u_m(r)|^2 r\, dr < \infty.$$

Now

$$\Delta u = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) u,$$

so from (2.1) we obtain from matching the coefficient of $e^{im\theta}$ in both sides that

$$\begin{aligned}
-\rho\omega^2 m^2 u_m &= T\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right) u_m \\
&\quad - E\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right)^2 u_m - i\gamma\omega m u_m + p_m, \qquad (2.5)\\
u_m &= \frac{\partial u_m}{\partial r} = 0, \qquad r = R, 1.
\end{aligned}$$

Let $\{\varphi_{m,n}(r)\}_{n=1}^{\infty}$ be a complete set of eigenfunctions for the eigenproblem

$$\begin{aligned}
-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right)\varphi_{m,n} &+ \frac{E}{T}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right)^2 \varphi_{m,n} \\
&= \lambda_{m,n}\varphi_{m,n}, \qquad R < r < 1, \qquad (2.6)\\
\varphi_{m,n} = \frac{\partial \varphi_{m,n}}{\partial r} &= 0, \qquad r = R, 1,
\end{aligned}$$

which are normalized by the condition

$$\int_R^1 |\varphi_{m,n}|^2 r dr = 1$$

and where

$$0 < \lambda_{m,1} \leq \lambda_{m,2} \leq \cdots \leq \lambda_{m,n} \leq \cdots .$$

Thus, $\lambda_{m,n}$ (respectively, $\varphi_{m,n}$) are critical values (respectively, critical points) of the functional

$$J_m(\varphi) = \int_R^1 \left[\left|\frac{\partial \varphi}{\partial r}\right|^2 + \frac{m^2}{r^2}|\varphi|^2 + \frac{E}{T}\left|\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) - \frac{m^2}{r^2}\varphi\right|^2\right] r\, dr$$

subject to the constraints that the real-valued functions, $\varphi(r)$, satisfy

$$\int_R^1 |\varphi|^2 r\, dr = 1,$$

$$\varphi(R) = \varphi(1) = 0,$$

$$\frac{\partial \varphi}{\partial r}(R) = \frac{\partial \varphi}{\partial r}(1) = 0.$$

It can be shown by classical arguments [6] that for $m = 0, \pm 1, \pm 2, \cdots$,

$$\lim_{n \to \infty} \lambda_{m,n} = +\infty. \tag{2.7}$$

We also note that $\lambda_{m,n}$ depends on R, i.e., $\lambda_{m,n} = \lambda_{m,n}(R)$ and that we can obtain from the representation of $\lambda_{m,n}$ as critical values of $J_m(\varphi)$ that $\lambda_{m,n}(R) \geq \lambda_{m,n}(0)$ for $0 \leq R < 1$. Now

$$\begin{aligned}
&\int_R^1 \left| \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) - \frac{m^2}{r^2} \varphi \right|^2 r\, dr \\
&= \int_R^1 \left[\left| \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) \right|^2 + \frac{m^4}{r^4} |\varphi|^2 - \frac{2m^2}{r^3} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) \varphi \right] r\, dr
\end{aligned}$$

and since φ satisfies the above boundary conditions it follows from integration by parts that

$$\begin{aligned}
&\int_R^1 \frac{m^2}{r^3} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) \varphi r\, dr \\
&= -m^2 \int_R^1 \frac{\partial \varphi}{\partial r} \frac{\partial}{\partial r} \left(\frac{\varphi}{r^2} \right) r\, dr \\
&= -m^2 \int_R^1 r^{-2} \left| \frac{\partial \varphi}{\partial r} \right|^2 r\, dr + 2m^2 \int_R^1 r^{-3} \frac{\partial \varphi}{\partial r} \varphi r\, dr \\
&= -m^2 \int_R^1 r^{-2} \left| \frac{\partial \varphi}{\partial r} \right|^2 r\, dr + 2m^2 \int_R^1 r^{-4} |\varphi|^2 r\, dr.
\end{aligned}$$

All the calculations in this paragraph can be used to show that

$$\begin{aligned}
J_m(\varphi) = &\int_R^1 \left[\left| \frac{\partial \varphi}{\partial r} \right|^2 + \frac{m^2}{r^2} |\varphi|^2 \right] r\, dr \\
&+ \frac{E}{T} \int_R^1 \left[\left| \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \varphi \right) \right|^2 + \frac{m^4 - 4m^2}{r^4} |\varphi|^2 + \frac{2m^2}{r^2} \left| \frac{\partial \varphi}{\partial r} \right|^2 \right] r\, dr
\end{aligned} \tag{2.8}$$

for φ satisfying the imposed boundary conditions.

Classical arguments [6] can also be used to show that

$$\int_R^1 \varphi_{m,n}(r) \varphi_{m,p}(r) r\, dr = \delta_{n,p}.$$

Further, if

$$\int_R^1 |v|^2 r\,dr < \infty,$$

then the expansion

$$v(r) = \sum_{n=1}^{+\infty} v_{m,n}\varphi_{m,n}(r)$$

where

$$v_{m,n} = \int_R^1 v(r)\varphi_{m,n}(r) r\,dr$$

has the properties that

$$\int_R^1 |v|^2 r\,dr = \sum_{n=1}^{+\infty} |v_{m,n}|^2$$

and

$$\int_R^1 \left| v(r) - \sum_{n=1}^{N} v_{m,n}\varphi_{m,n}(r)\right|^2 r\,dr = \sum_{n=N+1}^{+\infty} |v_{m,n}|^2 \to 0$$

as $N \to \infty$. Finally, since

$$J_m(\varphi) \geq \frac{E}{T}\int_R^1 \frac{(m^4 - 4m^2)}{r^4}|\varphi|^2 r\,dr$$

and since

$$J_m(\varphi) \geq \int_R^1 \frac{m^2}{r^2}|\varphi|^2 r\,dr,$$

we have that

$$\lambda_{m,n} \geq \lambda_{m,1} \geq \max\left(\frac{E}{T}(m^4 - 4m^2), m^2\right). \tag{2.9}$$

These properties of the eigenfunctions $\varphi_{m,n}(r)$ can be used to construct the expansion

$$p(r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} p_{m,n}\varphi_{m,n}(r)e^{im\theta} \tag{2.10}$$

where

$$p_{m,n} = \frac{1}{2\pi}\int_0^{2\pi}\int_R^1 p(r,\theta)\, e^{-im\theta}\varphi_{m,n}(r) r\,dr.$$

Note that

$$\int_0^{2\pi}\int_R^1 |p|^2 r\,dr\,d\theta = 2\pi \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} |p_{m,n}|^2 < \infty.$$

We will now show that there exist unique coefficients, $u_{m,n}$, such that

$$u(r,\theta) = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty} u_{m,n}\varphi_{m,n}(r)e^{im\theta} \tag{2.11}$$

is a mean-square integrable solution to (2.1). If we formally substitute the expansions (2.10) and (2.11) into (2.1) we obtain the result that

$$-\rho\omega^2 m^2 u_{m,n} = -T\lambda_{m,n}u_{m,n} - i\gamma\omega m u_{m,n} + p_{m,n}$$

or

$$u_{m,n} = [T\lambda_{m,n} - \rho\omega^2 m^2 + i\gamma\omega m]^{-1} p_{m,n}.$$

So,

$$|u_{m,n}|^2 = [(T\lambda_{m,n} - \rho\omega^2 m^2)^2 + \gamma^2\omega^2 m^2]^{-1}|p_{m,n}|^2.$$

We note that by (2.9)

$$(T\lambda_{m,n} - \rho\omega^2 m^2)^2 + \gamma^2\omega^2 m^2 \geq \begin{cases} T^2\lambda_{0,1}^2 & \text{if } m = 0, \\ \dfrac{\gamma^2 T}{2\rho} & \text{if } m \neq 0,\ T \leq 2\rho\omega^2, \\ \dfrac{T}{2} & \text{if } m \neq 0,\ T > 2\rho\omega^2. \end{cases}$$

Thus,

$$|u_{m,n}|^2 \leq C_1 |p_{m,n}|^2 \tag{2.12}$$

where

$$C_1^{-1} = \min\left(T^2\lambda_{0,1}^2, \frac{\gamma^2 T}{2\rho}, \frac{T}{2}\right).$$

We have thus shown that if $p(r,\theta)$ is mean-square integrable, then the formal solution to (2.1)

$$u(r,\theta) = \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{+\infty} u_{m,n}\varphi_{m,n}(r)e^{im\theta},$$

is unique and by (2.12) satisfies the estimate

$$\int_0^{2\pi}\int_R^1 |u(r,\theta)|^2 r\, dr\, d\theta \leq C_1 \int_0^{2\pi}\int_R^1 |p(r,\theta)|^2 r\, dr\, d\theta. \tag{2.13}$$

Since $\varphi_{m,n}$ satisfies the boundary conditions in (2.6), it follows that $u(r,\theta)$ satisfies (formally) the boundary conditions of (2.1).

Now it follows from (2.6) that $e^{im\theta}\varphi_{m,n}(r)$ are the eigenfunctions of

$$\begin{gathered} \left[-\Delta + \frac{E}{T}\Delta^2\right] e^{im\theta}\varphi_{m,n}(r) = \lambda_{m,n}e^{im\theta}\varphi_{m,n}(r), \\ e^{im\theta}\varphi_{m,n}(R) = e^{im\theta}\varphi_{m,n}(1) = 0, \\ \frac{\partial}{\partial r}[e^{im\theta}\varphi_{m,n}](R) = \frac{\partial}{\partial r}[e^{im\theta}\varphi_{m,n}](1) = 0. \end{gathered} \tag{2.14}$$

Thus, since by integration by parts

$$\int_0^{2\pi}\int_R^1 [T\nabla(e^{im\theta}\varphi_{m,n})\cdot\nabla(e^{iq\theta}\varphi_{q,p}) + E\Delta(e^{im\theta}\varphi_{m,n})\Delta(e^{iq\theta}\varphi_{q,p})]r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_R^1 [(-T\Delta + E\Delta^2)(e^{im\theta}\varphi_{m,n})]e^{iq\theta}\varphi_{q,p} r\,dr\,d\theta$$

$$= T\lambda_{m,n}\int_0^{2\pi}\int_R^1 (e^{im\varphi}\varphi_{m,n})(e^{iq\theta}\varphi_{q,p})r\,dr\,d\theta$$

$$= 2\pi T\lambda_{m,n}\delta_{m,q}\delta_{n,p},$$

we also have the bound

$$\int_0^{2\pi}\int_R^1 [T|\nabla u|^2 + E|\Delta u|^2]r\,dr\,d\theta = 2\pi T\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\lambda_{m,n}|u_{m,n}|^2$$

$$= 2\pi T\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\frac{\lambda_{m,n}}{[(T\lambda_{m,n}-\rho\omega^2 m^2)^2+\gamma^2\omega^2 m^2]}|p_{m,n}|^2 \tag{2.15}$$

$$\leq 2\pi C_2\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}|p_{m,n}|^2 = C_2\int_0^{2\pi}\int_R^1 |p(r,\theta)|^2 r\,dr\,d\theta$$

where by (2.7) and (2.9)

$$C_2 = \max_{m,n}\left[\frac{T\lambda_{m,n}}{(T\lambda_{m,n}-\rho\omega m^2)^2+\gamma^2 w^2 m^2}\right] < \infty.$$

Since

$$\int_0^{2\pi}\int_R^1 (-T\Delta + E\Delta^2)(e^{im\theta}\varphi_{m,n})\cdot(-T\Delta + E\Delta^2)(e^{iq\theta}\varphi_{q,p})r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_R^1 (T\lambda_{m,n}e^{im\theta}\varphi_{m,n})(T\lambda_{q,p}e^{iq\theta}\varphi_{q,p})r\,dr\,d\theta$$

$$= 2\pi T^2\lambda_{m,n}\lambda_{q,p}\delta_{m,q}\delta_{n,p},$$

we have the stronger bound

$$
\begin{aligned}
\int_0^{2\pi}\int_R^1 |-T\Delta u+E\Delta^2 u|^2 r\,dr\,d\theta &= 2\pi T^2 \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\lambda_{m,n}^2|u_{m,n}|^2\\
&= 2\pi \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\frac{(T\lambda_{m,n})^2}{(T\lambda_{m,n}-\rho\omega^2 m^2)^2+\gamma^2\omega^2 m^2}|p_{m,n}|^2\\
&\le 2\pi C_3 \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}|p_{m,n}|^2\\
&= C_3\int_0^{2\pi}\int_R^1 |p(r,\theta)|^2 r\,dr\,d\theta
\end{aligned}
\tag{2.16}
$$

where by (2.7) and (2.9)

$$C_3=\max\left[\frac{(T\lambda_{m,n})^2}{(T\lambda_{m,n}-\rho\omega^2m^2)^2+\gamma^2\omega^2m^2}\right]<\infty.$$

The inequalities (2.13), (2.15), and (2.16) can be used to show that if $p(r,\theta)$ is mean-square integrable, then the solution $u(r,\theta)$ to (2.1) that we have constructed has the property that all of its partial derivatives of order less than or equal to four are mean-square integrable [1]. Further, this implies that all of the partial derivatives of $u(r,\theta)$ of order less than or equal to two are continuous and that the boundary conditions are satisfied in the classical sense [1].

We review the above results by the following theorem.

THEOREM 2.1. *We suppose that $E_p>0$, $T>0$, $\gamma>0$, and $p\in L^2(\Omega)$. Then there exists a unique solution $u\in H_0^2(\Omega)\cap H^4(\Omega)$ to (2.1). Further, there exist positive constants C_1, C_2, and C_3 such that*

$$\int_\Omega u^2\,dx\le C_1\int_\Omega p^2\,dx, \tag{2.17}$$

$$\int_\Omega [T|\nabla u|^2+E|\Delta u|^2]\,dx\le C_2\int_\Omega p^2\,dx, \tag{2.18}$$

$$\int_\Omega |-T\Delta u+E\Delta^2u|^2\,dx\le C_3\int_\Omega p^2\,dx. \tag{2.19}$$

The constants C_1, C_2, and C_3 can be chosen independent of R. Also, the constant C_1, can be chosen independent of ω.

A more detailed analysis of $\lambda_{m,n}$ and the constant C_2 can be used to demonstrate that C_2 is independent of ω for $\rho\omega^2\le T$. However, we prefer to give the following elementary proof.

PROPOSITION 2.2. *Assume that*

$$\rho\omega^2\le T. \tag{2.20}$$

If $p \in L^2(\Omega)$, then there exists a unique solution to

$$
\begin{aligned}
&E\Delta^2 u - T\Delta u + \rho\omega^2 \frac{\partial^2 u}{\partial\theta^2} + \gamma\omega\frac{\partial u}{\partial\theta} = p \quad \text{in } \Omega, \\
&u \in H_0^2(\Omega).
\end{aligned} \tag{2.21}
$$

Moreover, there exists a constant $C_4 = C_4(E)$, independent of ω, such that

$$
|u|_{H_0^2(\Omega)} \le C_4 |p|_{L^2(\Omega)}. \tag{2.22}
$$

Proof. This is a straightforward application of the Lax–Milgram theorem. Indeed, consider the weak formulation of (2.21), i.e., set

$$
a(u,w) = \int_\Omega E\Delta u \cdot \Delta w + T\nabla u \cdot \nabla w - \rho\omega^2 \frac{\partial u}{\partial\theta}\cdot\frac{\partial w}{\partial\theta} + \gamma\omega\frac{\partial u}{\partial\theta}\cdot w\,dx.
$$

Then clearly $a(u,w)$ is a bilinear, continuous form on $H_0^2(\Omega)$. Moreover,

$$
\begin{aligned}
a(u,u) &= \int_\Omega E\Delta u \cdot \Delta u + T\nabla u \cdot \nabla u - \rho\omega^2 \left(\frac{\partial u}{\partial\theta}\right)^2 + \gamma\frac{w}{2}\frac{\partial}{\partial\theta}u^2\,dx \\
&= \int_\Omega E\Delta u \cdot \Delta u + T\nabla u \cdot \nabla u - \rho\omega^2 \left(\frac{\partial u}{\partial\theta}\right)^2 dx.
\end{aligned} \tag{2.23}
$$

(We used the fact that since $\Omega \subset \mathbb{R}^2$, $H_0^2(\Omega) \subset C(\bar{\Omega})$ and $u(r,\theta) = u(r,\theta+2\pi)$.) Now, recall that

$$
\begin{aligned}
\frac{\partial u}{\partial\theta} &= \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial\theta} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial\theta} \\
&= -r\sin\theta\frac{\partial u}{\partial x_1} + r\cos\theta\frac{\partial u}{\partial x_2}.
\end{aligned}
$$

Hence, by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\left(\frac{\partial u}{\partial\theta}\right)^2 &= r^2\left(-\sin\theta\frac{\partial u}{\partial x_1} + \cos\theta\frac{\partial u}{\partial x_2}\right)^2 \\
&\le |\nabla u|^2 \qquad \text{a.e. on } \Omega.
\end{aligned}
$$

Recalling (2.23) we obtain

$$
\begin{aligned}
a(u,u) &= \int_\Omega E\Delta u\cdot\Delta u + (T-\rho\omega^2)|\nabla u|^2\,dx \\
&\ge E\int_\Omega (\Delta u)^2\,dx.
\end{aligned}
$$

Since E is assumed to be positive, and since

$$\int_\Omega (\Delta u)^2\, dx$$

defined on $H_0^2(\Omega)$ is a norm equivalent to the usual one, $a(u,w)$ is a bilinear, continuous, coercive form on $H_0^2(\Omega)$. Now, for $p \in L^2(\Omega)$ it is clear that

$$w \longmapsto \int_\Omega pw\, dx$$

is a continuous linear form on $H_0^2(\Omega)$, so, by the Lax–Milgram theorem there is a unique u in $H_0^2(\Omega)$ satisfying

$$a(u,w) = \int_\Omega pw\, dx \quad \forall w \in H_0^2(\Omega).$$

Moreover, taking $w = u$ in the above equality, we can easily see that (2.22) holds.

It is easy to see that u satisfies (2.21) in the distributional sense. So, we have

$$\Delta^2 u \in L^2(\Omega)$$

since all the other functions appearing in (2.21) are in $L^2(\Omega)$. Hence, by well-known results $u \in H^4(\Omega)$. In particular we recover the fact that the condition

$$u = \frac{\partial u}{\partial r} = 0 \qquad \text{on } \partial\Omega$$

holds in the usual sense (see [1], [7] for details). □

We could have assumed p in the dual of $H_0^2(\Omega)$ and the existence of a weak solution to (2.21) would still have held true.

We note that (2.20) is the condition that (2.21) be elliptic when $E \equiv 0$. In the case that $\gamma \equiv 0$, existence and uniqueness can fail when $\rho\omega^2 > T$. If $\gamma \equiv 0$, then

$$(T\lambda_{m,n} - \rho\omega^2 m^2)u_{m,n} = p_{m,n}.$$

Hence, if $T\lambda_{m,n} = \rho\omega^2 m^2$ for some m,n, then there exist pressures, p, such that (2.21) does not have a solution. One example is clearly

$$p(r,\theta) = e^{im\theta}\varphi_{m,n}(r).$$

Also, in this case ($T\lambda_{m,n} = \rho\omega^2 m^2$ for some m,n) solutions to (2.21) are not unique since if $u(r,\theta)$ is a solution to (2.21), then

$$u(r,\theta) + e^{im\theta}\varphi_{m,n}(r)$$

is also a solution to (2.21). However, Theorem 2.1 guarantees existence and uniqueness when $\gamma > 0$ for all ω.

We note that for the floppy disk, the outer edge is not bonded to a rigid support disk. In this case, the tension depends on r. Furthermore, the radial tension coefficient vanishes at the outer edge [2]. Although clamped plate boundary conditions are appropriate at the inner edge, the plate is free at the outer edge [2]. The head in a floppy disk system does not fly above the medium on an air bearing. In this case, p in (1.1) represents the load on the disk from contact with the head. The analysis given for Proposition 2.2 applies to the floppy disk system if E is sufficiently large. In this case it is no longer true that $\rho\omega^2 \le T$ everywhere in Ω. However, for $E/\rho\omega^2$ sufficiently large, we can use the inequality

$$\rho\omega^2 \int_\Omega \left(\frac{\partial u}{\partial \theta}\right)^2 dx \le \frac{E}{2} \int_\Omega (\Delta u)^2\, dx$$

for $u \in H^2(\Omega)$ and $u = \dfrac{\partial u}{\partial n} = 0$ on the inner edge of the disk.

3. The coupled problem. The steady-state solution of (1.1), (1.4) is

$$E\Delta^2 u - T\Delta u + \rho\omega^2 \frac{\partial^2 u}{\partial\theta^2} + \gamma\omega\frac{\partial u}{\partial\theta} = p - p_a, \qquad \text{in } \Omega, \tag{3.1}$$

$$u \in H_0^2(\Omega),$$

$$\nabla\cdot(h(u)^3 p\nabla p) = 6\mu\mathbf{V}\cdot\nabla(ph(u)), \qquad \text{in} \Gamma, \tag{3.2}$$

$$p = p_a \qquad \text{on } \partial\Gamma,$$

where $\mathbf{V} = \omega(-x_2, x_1)$, $\quad \Omega = \{\, x = (x_1, x_2) \mid R < r < 1\,\}$.

We shall restrict p so that $p \geq 0$ and set $v = p^2$. The problem then becomes to find (u, v) such that

$$E\Delta^2 u - T\Delta u + \rho\omega^2 \frac{\partial^2 u}{\partial\theta^2} + \gamma\omega\frac{\partial u}{\partial\theta} = \sqrt{v} - \sqrt{v_a}, \qquad \text{in } \Omega, \tag{3.3}$$

$$u \in H_0^2(\Omega),$$

$$\nabla\cdot(h(u)^3\nabla v) = \omega\mathbf{W}\cdot\nabla(\sqrt{v}\quad h(u)), \qquad \text{in } \Gamma, \tag{3.4}$$

$$v = v_a, \quad \text{on } \partial\Gamma,$$

where we have set $v_a = p_a^2$, $\mathbf{W} = 12\mu(-x_2, x_1)$.

We are going to solve (3.4) in a weak sense. Noting that $\nabla\cdot\mathbf{W} = 0$ we see that if v satisfies (3.4)—say in a classical sense—we have

$$\begin{aligned} &\int_\Omega h(u)^3\nabla v\cdot\nabla\xi\,dx - \int_\Omega \omega\sqrt{v}\,h(u)\mathbf{W}\cdot\nabla\xi\,dx = 0 \quad \forall\xi \in H_0^1(\Omega), \\ &v = v_a \quad \text{on } \partial\Omega. \end{aligned} \tag{3.5}$$

From the Sobolev embedding theorem [1], [7] we have that

$$H_0^2(\Omega) \subset C(\bar{\Omega})$$

with continuous inclusion. So, for some positive constant C_5 we have that

$$|w|_{L^\infty(\Omega)} \leq C_5|w|_{H_0^2(\Omega)}, \qquad w \in H_0^2(\Omega).$$

Let us now collect our assumptions. We assume that the function φ (see (1.5)) satisfies

$$\begin{aligned} &\varphi \text{ is a Lipschitz continuous function on } \Gamma, \\ &0 < m \leq \varphi(x) \leq M \quad \text{a.e. for } x \in \Gamma, \end{aligned} \tag{3.6}$$

where m and M are two positive constants.

Assume also that

$$\Gamma \subset \Omega \text{ is a domain of } \mathbb{R}^2 \text{ with Lipschitz boundary;} \tag{3.7}$$

let $|\Gamma|$ denote the Lebesgue measure of Γ and let d denote the smallest width of a strip containing Γ. Then we can prove the following theorem.

THEOREM 3.1. *Let m' be any positive number such that*

$$0 < m' < m.$$

There exists a solution (u,v) of (3.3), (3.4) if (3.6) and (3.7) hold and if

$$\left(\frac{a+\sqrt{a^2+4b}}{2}\right) \leq p_a \frac{(m-m')}{C_4C_5} \tag{3.8}$$

where

$$a = \frac{[12d\omega\mu(M+m-m')]^2|\Gamma|^{1/2}}{(m')^6}, \qquad b = a|\Gamma|^{1/2}p_a^2.$$

Proof. Set

$$K_{\mathscr{R}} = \{\, v \in L^2(\Gamma) \mid v \geq 0 \text{ a.e. on } \Gamma,\ |v - v_a|_{L^2(\Gamma)} \leq \mathscr{R} \,\} \tag{3.9}$$

where $\mathscr{R}$ is a positive real number that we will choose later on. It is clear that $K_{\mathscr{R}}$ is a closed convex set of $L^2(\Gamma)$.

For $v \in K_{\mathscr{R}}$ we have

$$\sqrt{v} - \sqrt{v_a} \in L^2(\Omega) \tag{3.10}$$

(v is, of course, supposed to be extended by v_a outside of Γ). Indeed, this is an easy consequence of the inequality

$$|\sqrt{v} - \sqrt{v_a}| \leq \frac{1}{\sqrt{v_a}}|v - v_a|. \tag{3.11}$$

So, by Theorem 2.1 and Proposition 2.2 there exists a unique solution u of

$$\begin{aligned} &E\Delta^2 u - T\Delta u + \rho\omega^2\frac{\partial^2 u}{\partial\theta^2} + \gamma\omega\frac{\partial u}{\partial\theta} = \sqrt{v} - \sqrt{v_a}, \qquad \text{in } \Omega,\\ &u \in H_0^2(\Omega), \end{aligned} \tag{3.12}$$

and there is a constant $C_4 = C_4(E)$ such that

$$|u|_{H_0^2(\Omega)} \leq \frac{C_4}{\sqrt{v_a}}|v - v_a|_{L^2(\Gamma)} \tag{3.13}$$

(we use here the fact that, by (3.11), we have $|\sqrt{v} - \sqrt{v_a}|_{L^2(\Gamma)} \leq |v - v_a|_{L^2(\Gamma)}/\sqrt{v_a}$). The constant C_4 is independent of ω for $\rho\omega^2 \leq T$. Let m' be any positive number such that

$$0 < m' < m.$$

Let us show first that we can select $\mathscr{R}$ in such a way that

$$h(u) = \varphi - u \geq m' > 0. \tag{3.14}$$

Recall from the Sobolev embedding theorem given above that

$$|u|_{L^\infty(\Omega)} \le C_5 |u|_{H_0^2(\Omega)}$$
$$\le \frac{C_4 C_5}{\sqrt{v_a}} |v - v_a|_{L^2(\Gamma)} \qquad \text{(by (3.13))}.$$

Now, by (3.6), (3.14) will hold if

$$|u|_\infty \le m - m' \tag{3.15}$$

So (3.14) will hold if

$$\frac{C_4 C_5 \mathscr{R}}{\sqrt{v_a}} \le m - m'$$

or, equivalently,

$$\mathscr{R} \le \sqrt{v_a} \frac{(m - m')}{C_4 C_5}. \tag{3.16}$$

Assume that $\mathscr{R}$ has been chosen such that (3.16) holds. Then since $h(u)$ is strictly positive there exists a unique function $\mathscr{T}(v) \ge 0$ which is a solution of

$$\begin{aligned} &\int_\Gamma h(u)^3 \nabla \mathscr{T}(v) \cdot \nabla \xi \, dx - \omega \int_\Gamma \sqrt{\mathscr{T}(v)} h(u) \mathbf{W} \cdot \nabla \xi \, dx = 0, \qquad \forall \xi \in H_0^1(\Omega), \\ &\mathscr{T}(v) = v_a, \quad \text{on } \Gamma \end{aligned} \tag{3.17}$$

(we refer to [5] for a proof of this result).

If we can prove that $\mathscr{T}(v)$ has a fixed point we will be done. First, let us prove that for a suitable choice of $\mathscr{R}$, T maps $K_{\mathscr{R}}$ into itself. Indeed, we already know that $\mathscr{T}(v) \ge 0$. Next, if we take $\xi = \mathscr{T}(v) - v_a$ in (3.17) we get

$$(m')^3 |\nabla(\mathscr{T}(v) - v_a)|^2_{L^2(\Gamma)} \le \int_\Gamma h^3(u) |\nabla(\mathscr{T}(v) - v_a)|^2 \, dx$$
$$= \omega \int_\Gamma h(u) \sqrt{\mathscr{T}(v)} \mathbf{W} \cdot \nabla(\mathscr{T}(v) - v_a) \, dx.$$

Using the Cauchy–Schwarz inequality and recalling (3.6), (3.14) we deduce

$$(m')^3 |\nabla(\mathscr{T}(v) - v_a)|^2_{L^2(\Gamma)} \le 12\omega\mu(M + m - m') |\nabla(\mathscr{T}(v) - v_a)|_{L^2(\Gamma)} \left(\int_\Gamma \mathscr{T}(v) \, dx \right)^{1/2}$$

(note that $\sup_\Gamma |\mathbf{W}| \le 12\mu$). Hence, we have that

$$\begin{aligned} (m')^3 |\nabla(\mathscr{T}(v) - v_a)|_{L^2(\Gamma)} &\le 12\omega\mu(M + m - m') \left(\int_\Gamma \mathscr{T}(v) \, dx \right)^{1/2} \\ &\le 12\omega\mu(M + m - m') \left\{ \int_\Gamma |\mathscr{T}(v) - v_a| \, dx + |\Gamma| v_a \right\}^{1/2}. \end{aligned} \tag{3.18}$$

By the Poincaré inequality,

$$(m')^3|\mathscr{T}(v)-v_a|_{L^2(\Gamma)} \leq d(m')^3|\nabla(\mathscr{T}(v)-v_a)|_{L^2(\Gamma)}$$

$$\leq 12d\omega\mu(M+m-m')\{|\Gamma|^{1/2}|\mathscr{T}(v)-v_a|_{L^2(\Gamma)}+|\Gamma|v_a\}^{1/2} \tag{3.19}$$

where d denotes the smallest width of a strip containing Γ. So, we get

$$|\mathscr{T}(v)-v_a|^2_{L^2(\Gamma)} \leq a|\mathscr{T}(v)-v_a|_{L^2(\Gamma)}+b$$

with

$$a=\frac{[12d\omega\mu(M+m-m')]^2|\Gamma|^{1/2}}{(m')^6}, \qquad b=\frac{[12d\omega\mu(M+m-m')]^2|\Gamma|v_a}{(m')^6} \tag{3.20}$$

and $\mathscr{T}$ maps $K_{\mathscr{R}}$ into itself provided that

$$\left(\frac{a+\sqrt{a^2+4b}}{2}\right) \leq \mathscr{R}. \tag{3.21}$$

Assume that (see (3.16))

$$\left(\frac{a+\sqrt{a^2+4b}}{2}\right) < \sqrt{v_a}\frac{(m-m')}{C_4C_5} \tag{3.22}$$

then we can select $\mathscr{R}$ such that (3.16) and (3.21) hold. Thus for $v \in K_{\mathscr{R}}$, $\mathscr{T}(v) \in K_{\mathscr{R}}$. Now, from (3.18) it is clear that $\mathscr{T}(K_{\mathscr{R}})$ is relatively compact in $K_{\mathscr{R}}$ (since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, see [7]). So, provided we prove that $\mathscr{T}$ is continuous on $K_{\mathscr{R}}$, by the Schauder fixed point theorem (see [7]) we can conclude the existence of (u,v) satisfying (3.3), (3.5).

To prove the continuity of $\mathscr{T}$ we proceed as follows: let $v_n \in K_{\mathscr{R}}$ be such that $v_n \to v$ in $L^2(\Gamma)$. Let us denote by u_n the solution of (3.3) corresponding to $v=v_n$, and by u the one corresponding to v. From Theorem 2.1 and Proposition 2.2 we derive easily

$$|u_n-u|_{H^2_0(\Omega)} \leq C|(\sqrt{v_n}-\sqrt{v_a})-(\sqrt{v}-\sqrt{v_a})|_{L^2(\Gamma)}$$

$$\leq C|\sqrt{v_n}-\sqrt{v}|_{L^2(\Gamma)} \leq C\big|\sqrt{|v_n-v|}\big|_{L^2(\Gamma)}$$

$$=C\left(\int_\Gamma |v_n-v|\,dx\right)^{1/2} \leq C|\Gamma|^{1/4}|v_n-v|^{1/2}_{L^2(\Gamma)}.$$

(We used the Cauchy–Schwarz inequality.) Hence $u_n \to u$ in $H^2_0(\Omega)$ and also uniformly on $\bar{\Omega}$. (Recall that $H^2_0(\Omega) \subset C(\bar{\Omega})$ continuously.)

Now, from (3.19) we deduce that for some constant C independent of n we have

$$|\mathscr{T}(v_n)|_{H^1(\Gamma)} \leq C.$$

So we can extract a subsequence n_k from n such that

$$\mathscr{T}(v_{n_k}) \rightharpoonup w \quad \text{in } H^1(\Gamma), \qquad \mathscr{T}(v_{n_k}) \to w \quad \text{in } L^2(\Gamma). \tag{3.23}$$

If we let k go to $+\infty$ in the equality

$$\int_\Gamma h^3(u_{n_k})\nabla\mathscr{T}(v_{n_k})\cdot\nabla\xi - \omega\sqrt{\mathscr{T}(v_{n_k})}h(u_{n_k})\mathbf{W}\cdot\nabla\xi\,dx = 0 \qquad \forall\xi\in H_0^1(\Omega),$$

we obtain (recall that $h^3(u_{n_k}) \to h^3(u)$ uniformly)

$$\begin{aligned} &\int_\Gamma h^3(u)\nabla w\cdot\nabla\xi - \omega\sqrt{w}h(u)\mathbf{W}\cdot\nabla\xi\,dx = 0 \qquad \forall\xi\in H_0^1(\Omega),\\ &w = v_a \qquad \text{on } \partial\Gamma. \end{aligned} \tag{3.24}$$

By uniqueness of the solution of such a problem we have $w = \mathscr{T}(v)$ (see (3.17)). It results from (3.23) that the whole sequence $\mathscr{T}(v_n)$ converges toward $\mathscr{T}(v)$. This proves that $\mathscr{T}(v_n)$ converges toward $\mathscr{T}(v)$. Hence, $\mathscr{T}$ is continuous on $K_{\mathscr{R}}$ and this completes the proof of the theorem.

COROLLARY 3.1. *If ω or $|\Gamma|$ are small enough or m is large enough with M/m fixed and* (3.6) *and* (3.7) *hold, then there exists a solution (u, v) of* (3.3), (3.4).

Proof. Clearly, (3.8) holds for ω or $|\Gamma|$ small enough, all other quantities being kept fixed. Also, (3.8) holds if we set $m' = m/2$ and m is large enough with M/m fixed. □

It should be possible to apply the techniques of this paper to prove results similar to Theorem 3.1 for tape systems. Tape systems are usually modeled by a simplified shell model for the displacement of the tape and the compressible Reynolds lubrication equation for the air bearing [9].

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